Determining the Minimal Length Scale of the Generalized Uncertainty Principle from the Entropy-Area Relationship

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Abstract

We derive the formula of the black hole entropy with a minimal length of the Planck size by counting quantum modes of scalar fields in the vicinity of the black hole horizon, taking into account the generalized uncertainty principle (GUP). This formula is applied to some intriguing examples of black holes - the Schwarzschild black hole, the Reissner-Nordstrom black hole, and the magnetically charged dilatonic black hole. As a result, it is shown that the GUP parameter can be determined by imposing the black hole entropy-area relationship, which has a Planck length scale and a universal form within the near-horizon expansion.

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1 Introduction

There has been much interest in the end state of a small black hole after Hawking evaporation. In the context of thermodynamics based on the Bekenstein entropy [1] and the Hawking temperature [2], the black hole emits a radiation and it becomes smaller and hotter, which disappears when the evaporation ends. Then, the black hole will evaporate completely, leaving behind thermal radiation described by quantum-mechanical mixed states. Therefore, the information will be completely lost and the unitarity postulate of quantum theory may be broken, which is well-known as *the black hole information loss paradox*. Even though the black hole is charged with an electric and/or magnetic field, the situation is similar to the uncharged case. There exists a lower bound of the black hole mass called the extremal limit where the mass and the charge are in balance. As the small black hole radiates, it looses its mass and finally approaches the limit at which the black hole radiates no more.

However, this scenario is mainly based on the semi-classical analysis [3], assuming the classical background metric and disregarding the radiating energy compared to the rest energy of the black hole. Provided the black hole reaches the Planck size as it radiates, the emitted radiation energy is not any more negligible compared to the size of the black hole. Thus, when the size of the black hole is comparable to the Compton wavelength of the emitted radiation, the quantum fluctuation near the black hole affects the position of the black hole horizon, which leads to the breakdown of the semi-classical assumptions. Hence, we should include the back-reaction effect for the full analysis, which means that the complete quantum gravity is required in order to give a definite answer on the fate of the black hole.

Recently, it has been proposed that there might be a minimal length with the Planck scale, modifying usual commutation relations of the Heisenberg's uncertainty principle to the generalized uncertainty principle (GUP). The tensorial forms of the commutation relation [4, 5, 6, 7, 8] are

$$[x_i, p_j] = i\hbar(1 + \lambda p^2)\delta_{ij}, \tag{1.1}$$

$$[x_i, x_j] = 2i\hbar\lambda(p_i x_j - p_j x_i), \quad [p_i, p_j] = 0, \tag{1.2}$$

which leads to the minimal length uncertainty,

$$\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar}(\Delta p)^2. \tag{1.3}$$

It has been shown that the commutation relation implying the minimal length is not uniquely determined [4]. Indeed, its conceptual origin comes from the string theory [5], which resembles noncommutative geometry [6, 7]. In the context of string theory, the GUP provides the improved uncertainty relation (1.3) and the GUP parameter λ is determined as the fundamental constant associated with the string tension, $\lambda \simeq \alpha' \sim (10^{-32}\mathrm{cm})^2$, implying the existence of the minimal length with a Planck scale, $\Delta x \geq 2\sqrt{\alpha'} \sim 10^{-32}\mathrm{cm}$.

From these reasons, the GUP has drawn much attention in diverse aspects - the modification of dispersion relations [7, 8, 9], the black hole entropy without brick walls [10, 11, 12, 13, 14, 15, 16], the black hole remnants (BHR) as a possible resolution of *the information loss paradox* [17], and the primordial black hole remnants as a candidate of the cold dark matter (CDM) [18].

On the other hand, it has been shown that the GUP relation (1.3) can be derived from the model-independent ways from the quantum theory of gravitation [19], where the GUP parameter has not been specified. However, it can be determined through certain specific models and measurements such as a string theory or the full theory of quantum gravity.

One may think the GUP parameter should be determined by some physical laws or principles because the relation is derived from the basic assumptions of quantum theory of gravitation. Motivated by this, we would like to compute the entropy of scalar fields in the black hole background in the presence of the minimal length, and we show that the black hole entropy-area relationship can fix the scale of the minimal length of the GUP in Ref. [19]. More precisely, the generic metric ansatz with the spherical symmetry is taken into account and the entropy can be written in the form of the polynomial of the minimal length parameter in the near-horizon limit. As a result, the most dominant term describes the entropy that is proportional to the area of the event horizon while the subleading terms are quite negligible in the regime of the large black hole. The GUP parameter determined by the area law is universal up to the second order expansion of the near-horizon limit in some specific models, which is the order of the Planck scale.

In section 2, we shall derive the generic formula of the entropy of scalar fields in the background of the spherical symmetric black hole metric assuming semi-classical approximations and keeping the second order expansion of the near-horizon parameter. In section 3, the charged dilatonic black hole solutions with an arbitrary coupling between the dilaton and the U(1) gauge field strength are taken into account. For specific values of the coupling, the solution describes the Schwarzschild (SS), the Reissner-Nordstrom (RN), and the magnetically charged dilatonic black holes. In section 4, for these specific cases of the coupling, we show that the black hole entropy-area relationship can determine the GUP parameter as a Planck scale. Finally, some discussions are included in section 5.

2 Derivation of Entropy with the Minimal Length

Let us consider the Klein-Gordon equation for a massive scalar field in the background of the black hole $(\Box - \mu^2)\Phi = 0$, where $\Box = \nabla_\mu \nabla^\mu$ and μ is the mass of the scalar field. Using an ansatz $\Phi = \Psi(r, \theta, \varphi)e^{-i\omega t}$, then the field equation becomes

$$\Psi'' + \left(\frac{f'}{f} + 2\frac{R'}{R}\right)\Psi' + \frac{1}{f}\left[\frac{\omega^2}{f} - \mu^2 + \frac{1}{R^2}(\partial_\theta^2 + \cot\theta\partial_\theta + \csc^2\theta\partial_\varphi^2)\right]\Psi = 0, \tag{2.1}$$

where the prime denotes d/dr, and f and R^2 are the metric functions of the spherically symmetric metric,

$$(ds)^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + R^2(r)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.2}$$

Assuming the Wenzel-Kramers-Brillouin (WKB) approximation [20] with $\Psi \simeq e^{iS(r,\theta,\varphi)}$, it is found to be $p_r^2 = \frac{1}{f}\left[\frac{\omega^2}{f} - \mu^2 - \frac{p_\theta^2}{R^2} - \frac{p_\varphi^2}{R^2\sin^2\theta}\right]$, where $p_r = \partial S/\partial r$, $p_\theta = \partial S/\partial\theta$, and $p_\varphi = \partial S/\partial\varphi$. We have the squared module of momentum given by $p^2 = g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + g^{\varphi\varphi}p_\varphi^2 = \omega^2/f - \mu^2$ and the volume in the momentum phase space is

$$V(r,\theta) = \int dp_r dp_\theta dp_\varphi = \frac{4}{3}\pi\frac{R^2(r)}{\sqrt{f}}\sin\theta\left(\frac{\omega^2}{f} - \mu^2\right)^{3/2} \tag{2.3}$$

with $\omega \geq \mu\sqrt{f}$. The number of quantum states are given by the weighted phase space volume measure [8],

$$n(\omega) \;\; = \;\; \frac{1}{(2\pi)^3}\int dr d\theta d\varphi dp_r dp_\theta dp_\varphi \frac{1}{(1+\lambda p^2)^3}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^3}\int dr d\theta d\varphi \frac{V(r,\theta)}{(1+\lambda p^2)^3} \\
&= \frac{2}{3\pi}\int dr \frac{R^2(r)\left(\frac{\omega^2}{f}-\mu^2\right)^{3/2}}{\sqrt{f}\left[1+\lambda\left(\frac{\omega^2}{f}-\mu^2\right)^3\right]}.
\end{aligned} \tag{2.4}
$$

One might think that arbitrary integration measures can be chosen without deforming the commutation relations Eqs. (1.1) and (1.2). Specifically, the measure in Eq. (2.4) may be absorbed in a suitable rescaling of the fields and in a suitable redefinition of the operators that act on the fields. However, this is not the case since this measure is nontrivial in that the measure of the phase space corresponding to the number of density of states should be consistently derived as long as we follow the Liouville theorem as indicated in Ref. [8], where this weighted volume element should be invariant under the infinitesimal time translations. Actually, the time evolution is subjected to the GUP through modified Hamiltonian equations of motion. Of course, taking $\lambda = 0$, then the original density of states is naturally recovered. The improved measure seems to be plausible in the sense that the minimal length plays a role of the ultraviolet cut-off which is naturally introduced in the denominator in Eq. (2.4). On the other hand, the construction of the Hilbert space representation with the GUP has been already done in Ref. [7], although it is no longer unique. Indeed, the representation describing the minimal length uncertainty is for the momentum space. Taking into account it in a position space, the position eigenstates are in general no longer orthogonal unlike the momentum eigenstates. Of course, there might be diagonalizable but have no physical eigenstates since they are described on lattices in the position space. In this sense it looks difficult to construct the representation in the position space rather than that in the momentum space. However, for given commutation relations, the weighted volume factor of the phase space is uniquely determined and is independent of the choice of representation. From these reasons, the invariant phase space volume gives some corrections to the physical quantities through the minimal length uncertainty principle.

Now, taking into account a thin-layer around the event horizon of the black hole between r_+ and $r_+ + \epsilon$, where ϵ is an infinitesimal distance from the horizon, then the metric functions around

this layer are expanded as

$$f(r) \simeq \kappa(r-r_+) + f_2(r-r_+)^2, \quad R^2(r) \simeq r_0 + r_1(r-r_+) + r_2(r-r_+)^2, \tag{2.5}$$

where κ is a surface gravity with $\kappa = f'(r_+)$ and the other coefficients are $f_2 = f''(r_+)/2,\ r_0 = R^2(r_+),\ r_1 = [R^2(r_+)]'$, and $r_2 = [R^2(r_+)]''/2$ by keeping the second order of the expansion. Now we want to identify the proper length between the layer with the GUP minimal length. Then the GUP parameter associated with the minimal length $x_{min} = 2\sqrt{\lambda}$ is determined by

$$2\sqrt{\lambda} = \int_{x_+}^{x_++\epsilon} \frac{d\hat{x}}{\sqrt{f(\hat{x})}} \simeq \frac{\sqrt{\epsilon}}{\sqrt{\kappa}}\left(2 - \frac{f_2\epsilon}{3\kappa}\right), \tag{2.6}$$

which can be expressed in the alternate form of $\epsilon \simeq \kappa\lambda + \mathcal{O}(\lambda^2)$. From the free energy defined by

$$F = -\int_0^\infty d\omega \frac{n(\omega)}{e^{\beta\omega}-1}, \tag{2.7}$$

where β is the inverse Hawking temperature, the entropy, $\mathcal{S}_{BH} = \beta^2 \partial F/\partial\beta$, is straightforwardly calculated as

$$\begin{aligned}\mathcal{S}_{BH} &= \frac{\beta^3}{12\pi\lambda^3}\int_0^\infty dx \frac{x^4}{\sinh^2 x}\int_{r_+}^{r_++\epsilon} dr \frac{R^2(r) f}{(x^2+Bf)^3} \\ &\equiv \frac{\beta^3}{12\pi\lambda^3}\int_0^\infty dx \frac{x^4}{\sinh^2 x}\mathcal{I}(x),\end{aligned} \tag{2.8}$$

where $B = \beta^2/4\lambda$ and $x \equiv \beta\omega/2$ by setting $\mu = 0$ (massless scalar field) for simplicity. Since we have $R^2 f = r_0\kappa(r-r_+) + (r_1\kappa + r_0 f_2)(r-r_+)^2 + (r_2\kappa + r_1 f_2)(r-r_+)^3 + \mathcal{O}(r-r_+)^4$, the radial integration, $\mathcal{I}(x)$ in Eq. (2.8) is expressed by three parts near the horizon and expanded in terms of ϵ by keeping the ϵ^4-order terms,

$$\mathcal{I}(x) = \int_0^\epsilon d\hat{\epsilon} \frac{r_0\kappa\hat{\epsilon} + (r_1\kappa + r_0 f_2)\hat{\epsilon}^2 + (r_2\kappa + r_1 f_2)\hat{\epsilon}^3}{(x^2 + B\kappa\hat{\epsilon} + Bf_2\hat{\epsilon}^2)^3} \simeq \frac{a}{x^6} - \frac{b}{x^8} + \frac{c}{x^{10}}, \tag{2.9}$$

where

$$a = \frac{1}{2}r_0\kappa\epsilon^2 + \frac{1}{3}(r_1\kappa + r_0 f_2)\epsilon^3 + \frac{1}{4}(r_2\kappa + r_1 f_2)\epsilon^4, \tag{2.10}$$

$$b = \frac{\beta^2\kappa\epsilon^3}{16\lambda}\left[4r_0\kappa + 3(2r_0 f_2 + r_1\kappa)\epsilon\right], \quad c = \frac{3r_0\kappa^3\beta^4\epsilon^4}{32\lambda^2}. \tag{2.11}$$

Therefore, the black hole entropy becomes

$$\mathcal{S}_{BH} = \frac{\beta^3}{12\pi\lambda^3}\left[a\int_0^\infty \frac{dx}{x^2\sinh^2 x} - b\int_0^\infty \frac{dx}{x^4\sinh^2 x} + c\int_0^\infty \frac{dx}{x^6\sinh^2 x}\right]. \tag{2.12}$$

Since the integrations with respect to x can be regarded as a contour integration on a complex plane, we use the residue theorem and find

$$\int_0^\infty \frac{dx}{x^2 \sinh^2 x} = \frac{2}{\pi^2}\zeta(3), \quad \int_0^\infty \frac{dx}{x^4 \sinh^2 x} = -\frac{4}{\pi^4}\zeta(5), \quad \int_0^\infty \frac{dx}{x^6 \sinh^2 x} = \frac{6}{\pi^6}\zeta(7), \tag{2.13}$$

where $\zeta(n)$ is a zeta function. Plugging these into Eq. (2.12), then the black hole entropy is

$$\mathcal{S}_{BH} = \frac{\beta^3}{12\pi^3\lambda^3}\left[2a\zeta(3) + \frac{4b}{\pi^2}\zeta(5) + \frac{6c}{\pi^4}\zeta(7)\right], \tag{2.14}$$

which is the general formula of the entropy. Since the precise form of the black hole entropy depends on the specific metric, we shall apply this to some concrete black hole models in the following section.

3 Dilatonic Charged Black Holes with an Arbitrary Coupling

Now let us consider a four-dimensional low-energy dilaton gravity action with an arbitrary coupling from string theory given by [21]

$$I = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left[\mathcal{R} - 2(\nabla\phi)^2 - e^{-2\alpha\phi}F^2\right], \tag{3.1}$$

where G_N is a Newton's constant, ϕ is a dilaton field, F is a Maxwell field strength of a $U(1)$ subgroup of $E_8 \times E_8$ or Spin(32)/Z_2, and α is a coupling constant between dilaton and the Maxwell field strength. The charged dilatonic black hole solution with a spherical symmetry is given in the form of

$$f(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}, \quad R^2(r) = r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \tag{3.2}$$

$$e^{-2\alpha\phi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \quad F = Q\sin\theta d\theta \wedge d\varphi \tag{3.3}$$

where r_+ and r_- are related to the mass M and the magnetic charge Q of black holes as $2M = r_+ + \frac{1-\alpha^2}{1+\alpha^2}r_-$, $Q^2 = \frac{r_+ r_-}{1+\alpha^2}$, respectively. It is easy to verify that the action (3.1) has an electromagnetic dual symmetry under $\phi \to -\phi$ along with the fixed metric by defining the electric field strength as $\tilde{F}_{\mu\nu} = \frac{1}{2}e^{-2\alpha\phi}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$. The solution for $\alpha = 0$ describes the Ressiner-Nordstrom (RN) black hole while the one for $\alpha = 1$ corresponds to the magnetically charged black hole [22].

Moreover, the case of $\alpha = 0$ and $r_- = 0$ describes the Schwarzschild black hole. Note that the extremal limit $r_+ = r_-$ leads to $M^2 = Q^2/(1+\alpha^2)$. The Hawking temperature for non-extremal black holes is given by $T_H = \beta^{-1} = \frac{1}{2\pi r_+}\left(\frac{r_+ - r_-}{r_+}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}$, which implies that it always vanishes for the extremal limit unless $\alpha = 1$.

4 Universal Minimal Length Scale from the Entropy-Area Relationship

4.1 Schwarzschild (SS) and Reissner-Nordstrom (RN) black holes

For the SS black hole ($\alpha = 0$ and $r_- = 0$), the coefficients of the series expansion in Eq. (2.5) can be identified with $\kappa = 1/r_+$, $f_2 = -1/r_+^2$, $r_0 = r_+^2$, $r_1 = 2r_+$, and $r_2 = 1$ and then Eqs. (2.10) and (2.11) become

$$a = \frac{r_+\epsilon^2}{2} + \frac{\epsilon^3}{3} - \frac{\epsilon^4}{4r_+}, \quad b = \frac{\beta^2\kappa\epsilon^3 r_+}{4\lambda}, \quad c = \frac{3\beta^4\epsilon^4}{32\lambda^2 r_+}. \tag{4.1}$$

Plugging these into Eq. (2.14), one finds the black hole entropy in the polynomial form with respect to λ as

$$\mathcal{S}_{BH}^{SS} = \frac{A_H}{4G_N} + \frac{4}{9}\zeta(3) + \mathcal{O}(\lambda), \tag{4.2}$$

where A_H is the area of S^2-sphere at the event horizon, $A_H = 4\pi r_+^2$, and the GUP parameter is determined to $\lambda = 2G_N(\zeta(3) + 4\zeta(5) + 9\zeta(7))/3\pi$. Note that the GUP parameter depends on the Newton's constant of the Planck length scale since $\lambda \sim 3.061061275 \times G_N$. The leading order describes the area law of the black hole entropy with the $1/\lambda$-order contribution while the next leading term is the λ^0-th and higher order terms. However, in the large black hole limit we considered ($r_+ >> 1$), the subleading terms are quite negligible compared to the area term.

On the other hand, for the RN black hole, the coefficients for the expansion of the metric functions can be given by to $\kappa = (r_+ - r_-)/r_+$, $f_2 = -(r_+ - 2r_-)/r_+^3$, $r_0 = r_+^2$, $r_1 = 2r_+$, and $r_2 = 1$ and one can easily find

$$\begin{aligned} a &= \frac{(r_+ - r_-)\epsilon^2}{2} + \frac{\epsilon^3}{3} - \frac{(r_+ - 3r_-)\epsilon^4}{4r_+^2}, \\ b &= \frac{\beta^3\kappa\epsilon^3}{8\lambda}\left(2(r_+ - r_-) + \frac{3r_-\epsilon}{r_+}\right), \ c = \frac{3\beta^4\epsilon^4(r_+ - r_-)^3}{32\lambda^2 r_+^4}. \end{aligned} \tag{4.3}$$

Therefore, one can show the entropy of the RN black hole up to the zeroth order of λ,

$$\mathcal{S}_{BH}^{RN} = \frac{A_H}{4G_N} + \frac{4}{9}\zeta(3) + \frac{4r_-}{r_+}\zeta(5) + \mathcal{O}(\lambda), \tag{4.4}$$

where the same GUP parameter λ as the case of the SS black hole was chosen. The first term represents the area law of the black hole entropy, which agrees with the result in Ref. [15] while the higher order corrections are negligible for the large black hole case.

4.2 Magnetically charged dilatonic black hole

The magnetically charged dilatonic black hole solution is obtained when $\alpha = 1$, which yields $f(r) = 1 - r_+/r$ and $R^2(r) = r(r - r_-)$. For this metric solution, the coefficients of the series expansion near the horizon are easily obtained as $\kappa = 1/r_+,\ f_2 = -1/r_+,\ r_0 = r_+(r_+ - r_-),\ r_1 = 2r_+ - r_-,\ r_2 = 1$, and the coefficients (2.10) and (2.11) are found to be

$$\begin{aligned} a &= \frac{(r_+ - r_-)\epsilon^2}{2} + \frac{\epsilon^3}{3} - \frac{(r_+ - r_-)\epsilon^4}{4r_+^2}, \\ b &= \frac{\beta^2\epsilon^3}{16\lambda r_+}\left(4(r_+ - r_-) + \frac{3r_-\epsilon}{r_+}\right), \quad c = \frac{3\beta^4\epsilon^4(r_+ - r_-)}{32\lambda^2 r_+^2}. \end{aligned} \tag{4.5}$$

Hence, the entropy of the charged dilatonic black hole is found by keeping the λ^0-th order,

$$\mathcal{S}_{BH}^{\alpha=1} = \frac{A_H}{4G_N} + \frac{4}{9}\zeta(3) + \frac{2r_-}{r_+}\zeta(7) + \mathcal{O}(\lambda), \tag{4.6}$$

where $A_H = 4\pi r_+(r_+ - r_-)$ and $\lambda = 2G_N(\zeta(3) + 4\zeta(5) + 9\zeta(7))/3\pi$. Note that the first leading term represents the area of S^2-sphere at the horizon while the next subleading term is also negligible for the limit of $r_+ >> 1$. Therefore, the GUP parameter of the Planck scale has an universal form, as seen in the previous two cases up to the second order for the near-horizon expansion.

5 Discussion

We have derived the generic formula of the entropy of black holes by integrating quantum modes of scalar fields, taking into account the modified dispersion relation from the GUP and including the next subleading terms of the near-horizon expansion. This is a general result in that it is independent of the metric solutions as long as we assume the spherical symmetry. Since we

identified the ultra-violet cut-off ϵ with the GUP parameter λ, one can expand the metric function with respect to this GUP parameter and the expansion is also valid with respect to the minimal length of the Planck scale. The generic formula have been applied to the Schwarzschild black hole, the RN black hole, and the magnetically charged dilatonic black hole. The leading term of the entropies in three cases is the λ^{-1}-order, which clearly describes the area law of the black hole, whereas the subleading terms are negligible compared to the leading term since we have used the semi-classical assumptions that is valid only for the large black hole case. Especially, the constant contribution to the entropy can be removed by an appropriate normalization.

A short glance of three exemplified results reveals that the scale of the GUP parameter can be determined by the entropy-area relationship, which has the Planck length scale. Thus the unpredictable GUP parameter in Ref. [19] can be fixed by the black hole entropy-area relationship, which is universal up to the second order expansion of the near-horizon limit. Therefore, we conclude that the physical scale of the GUP parameter derived from the quantum theory of gravitation in Ref. [19] can be predicted by the black hole entropy-area relationship as $\lambda = \frac{2}{3\pi}(\zeta(3) + 4\zeta(5) + 9\zeta(7))G_N \sim G_N$, which has a Planckian scale of the minimal length, $\Delta x = 2\sqrt{\lambda} \sim 10^{-33}$ cm.

The minimal length uncertainty has been naturally derived from various contexts such as the string theory and the quantum gravity, which advocate the fundamental features of the UV/IR correspondence. One may expect that there is no invariance under the Lorentz transformation at the Planck scale where the minimal length uncertainty is dominant. In fact, the Lorentz symmetry breaking at the string scale or the Planck scale has been suggested in Ref. [23]. Furthermore, a similar possibility appears in the formalism of the κ-deformed Poincare group [24] since this quantum deformation closely related to the GUP also breaks the Lorentz invariance [25]. The minimal length is of great interest since it exhibits some intriguing feature of the UV/IR relation in a variety of contexts such as the AdS/CFT correspondence [26], noncommutative field theories [27], quantum gravity in asymptotically de Sitter space [28], and so on. In spite of these nice non-relativistic arguments, it should be possible to obtain the Lorentz covariant formulation of the GUP, which is unfortunately not successful up to now. For instance, the GUP commutation relations (1.1)

and (1.2) are not fully tensorial forms, which clearly breaks the Lorentz covariance. So, one may try to single out a preferred frame from a Lorentz covariant formulation. For this purpose, let us simply write the GUP relations in a Lorentz covariant fashion, $[x_\mu, p_\nu] = i\hbar(1 + \lambda p_\alpha p^\alpha)\eta_{\mu\nu}$, $[x_\mu, x_\nu] = 2i\hbar\lambda(p_\mu x_\nu - x_\mu p_\nu)$, $[p_\mu, p_\nu] = 0$. Then, the usual GUP commutation relations are recovered by dropping term $(p^0)^2$ from $p_\alpha p^\alpha$. However, this procedure is not equivalent to the non-relativistic limit [29] although the commutative limit can be well-defined for $\lambda = 0$. It implies that the above Lorentz covariant formulation fails so that the preferred frame giving the minimal length cannot be found from this naive formulation. Historically, as the relativistic quantum mechanics from the old quantum mechanics is not straightforward, it seems that it is not easy to achieve the relativistic formulation of the GUP which is the nontrivial extension of the quantum mechanics. We hope that this intriguing and important problem will be studied elsewhere.

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